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MULTIVARIATE DEPENDENT RELEVATIONS WITH APPLICATIONS, AND RELAT--ETC(U)

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MULTIVARIATE DEPENDENT RELEVATIONS  
WITH APPLICATIONS, AND RELATED TOPICS

by

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Abstract

"Relevation" is the name given by Krakowski (1973) (*Rev. Française Automat. Inform. Recherche Op., 7, Ser. V-2, 108-120*) to the distribution of failure time of a replacement from an aging stock. The present authors (Johnson and Kotz (1979, 1981) (*IEEE Trans. Reliability, R-28, 292-299; American Journal of Mathematical and Management Sciences, 1(2)*)) have extended this concept to include (i) hierachal replacement systems and (ii) dependence between lifetimes of original and replacement items. In this paper, we present some further developments, including first steps towards a synthesis of (i) and (ii).

Key Words and Phrases: Relevation; Dependence; Aging in Stock; Aging in Service; Farlie-Gumbel-Morgenstern Distribution; Hierachal Systems.

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### 1. Introduction.

The concept of a "relevation" was introduced by Krakowski (1973) to correspond to replacement from aging stocks. Suppose that when an element with survival distribution function (SDF)  $S_1(t)$  fails, it is replaced by an element with SDF  $S_2(t)$  taken from stock. If failure occurs at time  $t_1$ , then the conditional SDF of the replacement element is

$$S_{2|1}(t|t_1) = S_2(t)/S_2(t_1) ,$$

and the SDF of time of failure ( $T$ ) of the replacement element is

$$S_T(\tau) = S_1(\tau) + S_2(\tau) \int_0^\tau f_1(t)\{S_2(t)\}^{-1} dt , \quad (1)$$

where  $f_1(t) = -dS_1(t)/dt$  is the probability density function (PDF) corresponding to  $S_1(t)$ .

$S_T(\tau)$  is called the *relevation* of  $S_1(t)$  with  $S_2(t)$ .

This concept has been extended to situations wherein there is a hierachal system of replacement, and wherein the lifetimes of elements are not independent (Johnson and Kotz (1979, 1981)). In the present paper, we study combinations of extensions of this kind, and also give examples in which differential rates of aging in storage and in service are allowed for.

### 2. Dependent Relevations in a Hierachal System.

Consider a set of  $n$  elements  $\{\xi_1\}$ , with a simple replacement element  $\xi_2$ . This might occur, for example, when  $\xi_2$  is called into service as a back-up only when there is a failure among  $\{\xi_1\}$ . This system (comprised of  $\{\xi_1\}$  and  $\xi_2$ ) would quite possibly be only a small part of a complete hierachal system; study of such relatively simple systems is an essential prerequisite to study of more elaborate systems of which they could be a part.

We denote the failure times of the elements of  $\{\xi_1\}$  by  $T_{11}, T_{12}, \dots, T_{1n}$ , and of  $\xi_2$  by  $T_2$ . The joint survival distribution (SDF) of these  $(n+1)$  failure times is

$$S_{12}(t_1; t_2) = P\left(\bigcap_{j=1}^n (T_{1j} > t_{1j}) \cap (T_2 > t_2)\right) \quad (2)$$

( $\underline{t}_1 = (t_{11}, \dots, t_{1n})$ ). The joint SDF of  $T_1 = \min(T_{11}, \dots, T_{1n})$  -- the failure time of the first member of  $\{\xi_1\}$  to fail -- is

$$S_{12}(t_1; t_2) = S_{12}(\underline{t}_1 \underline{1}'; t_2)$$

where  $\underline{1}' = (1, 1, \dots, 1)$ ; and the SDF of  $T_1$  is

$$S_1(t_1) = S_{12}(\underline{t}_1 \underline{1}'; 0)$$

The SDF of the replacement element is then

$$S_T(\tau) = S_1(\tau) + \int_0^\tau f_1(t) \frac{S_{2|1}(\tau|t)}{S_{2|1}(t|t)} dt, \quad (5)$$

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where  $f_1(t) = -dS_1(t)/dt$  and  $S_{2|1}(\tau|t) = P(T_2 > \tau | T_1 = t)$ .

In general, explicit evaluation of (5) leads to rather complex formulas. As a compromise between realism and simplicity, we will (similarly as in Johnson and Kotz (1981)) use a generalized Farlie-Gumbel-Morgenstern distribution (Johnson and Kotz (1975)) as a special case for detailed examination. We suppose

$$S_{12}(\xi_1; t_2) = \{S_2(t_2) \prod_{j=1}^n S(t_{1j})\} [1 + \alpha \{1 - S_2(t_2)\} \sum_{j=1}^n \{1 - S(t_{1j})\} \\ + \beta \sum_{j=1}^{n-1} \sum_{j < j'} \{1 - S(t_{1j})\} \{1 - S(t_{1j'})\}] . \quad (6)$$

The parameters  $\alpha$  and  $\beta$  have to satisfy certain conditions for the distribution to be a proper SDF. These conditions can be summarized as

$$|(2r-n)\alpha| - \frac{1}{2}\{(2r-n)^2 - n\}\beta \leq 1 \quad (r = 0, 1, \dots, n) ,$$

or equivalently, (7)

$$r'|\alpha| - \frac{1}{2}(r'^2 - n)\beta \leq 1 \quad \begin{cases} r' = 0, 2, \dots, n & \text{for } n \text{ even} \\ r' = 1, 3, \dots, n & \text{for } n \text{ odd} \end{cases} .$$

The marginal SDF of each  $T_{1j}$  is  $S(t)$ ; the marginal SDF of  $T_2$  is  $S_2(t)$ . The joint SDF of  $T_{11}, \dots, T_{1n}$  is

$$S_{T_1}(\xi_1) = \{ \prod_{j=1}^n S(t_{1j}) \} [1 + \beta \sum_{j=1}^{n-1} \sum_{j < j'} \{1 - S(t_{1j})\} \{1 - S(t_{1j'})\}] , \quad (8)$$

which is an exchangeable distribution. This is likely to be a reasonable assumption if there are no systematic differences among the conditions to which the  $n$  elements of  $\{\xi_1\}$  are exposed.

The SDF of  $T_1$  (time to first failure in  $\{\xi_1\}$ ) is

$$S_1(t_1) = \{S(t_1)\}^n [1 + \frac{1}{2}n(n-1)\beta\{1 - S(t_1)\}^2] , \quad (9)$$

and the corresponding density function (PDF) is

$$f_1(t_1) = nf(t_1)\{S(t_1)\}^{n-1} [1 + \frac{1}{2}(n-1)\beta\{n - 2(n+1)S(t_1) + (n+2)[S(t_1)]^2\}]$$

$$= nf(t_1)\{S(t_1)\}^{n-1} [1 + \frac{1}{2}(n-1)\beta\{1 - S(t_1)\}\{n - (n+2)S(t_1)\}] . \quad (10)$$

The joint SDF of  $T_1$  and  $T_2$  is

$$S_{12}(t_1, t_2) = S_2(t_2)\{S(t_1)\}^n [1 + n\alpha\{1 - S(t_1)\}\{1 - S_2(t_2)\}]$$

$$+ \frac{1}{2}n(n-1)\beta\{1 - S(t_1)\}^2] , \quad (11.1)$$

with joint PDF

$$f_{12}(t_1, t_2) = nf_1(t_1)f_2(t_2)\{S(t_1)\}^{n-1} [1 + \alpha\{n - (n+1)S(t_1)\}\{1 - 2S_2(t_2)\}]$$

$$+ \frac{1}{2}(n-1)\beta\{1 - S(t_1)\}\{n - (n+2)S(t_1)\}] . \quad (11.2)$$

The conditional PDF of  $T_2$ , given  $T_1$ , is

$$f_{2|1}(t_2|t_1) = \frac{f_{12}(t_1, t_2)}{f_1(t_1)}$$

$$= f_2(t_2) \frac{1 + \alpha\{n - (n+1)S(t_1)\}\{1 - 2S_2(t_2)\} + \frac{1}{2}(n-1)\beta\{1 - S(t_1)\}\{n - (n+2)S(t_1)\}}{1 + \frac{1}{2}(n-1)\beta\{1 - S(t_1)\}\{n - (n+2)S(t_1)\}} \quad (12.1)$$

and the conditional SDF is

$$S_{2|1}(t_2|t_1) = S_2(t_2) \frac{1 + \alpha\{n - (n+1)S(t_1)\}\{1 - S_2(t_2)\} + \frac{1}{2}(n-1)\beta\{1 - S(t_1)\}\{n - (n+2)S(t_1)\}}{1 + \frac{1}{2}(n-1)\beta\{1 - S(t_1)\}\{n - (n+2)S(t_1)\}} \quad (12.2)$$

Hence from (5) the SDF of time to failure ( $T$ ) of the  $\xi_2$  (replacement) element is

$$\begin{aligned}
 S_T(\tau) &= \{S(\tau)\}^n [1 + \frac{1}{2}n(n-1)\beta\{1 - S(\tau)\}^2] \\
 &+ n \int_0^\tau f(t)\{S(t)\}^{n-1} [1 + \frac{1}{2}(n-1)\beta\{1 - S(t)\}\{n - (n+2)S(t)\}] \\
 &\times \frac{S_2(\tau)}{S_2(t)} \\
 &\times \frac{1 + \alpha\{1 - S_2(\tau)\}\{n - (n+1)S(t_1)\} + \frac{1}{2}(n-1)\beta\{1 - S(t_1)\}\{n - (n+2)S(t_1)\}}{1 + \alpha\{1 - S_2(t)\}\{n - (n+1)S(t_1)\} + \frac{1}{2}(n-1)\beta\{1 - S(t)\}\{n - (n+2)S(t)\}} dt
 \end{aligned} \quad (13.1)$$

As a further specialization, we will take  $S_2(t) = S(t)$ . Then making the substitution  $s = S(t)$  in the integral leads to

$$\begin{aligned}
 S_T(\tau) &= \{S(\tau)\}^n [1 + \frac{1}{2}n(n-1)\beta\{1 - S(\tau)\}^2] \\
 &+ nS(\tau) \int_{S(\tau)}^1 s^{n-2} [1 + \frac{1}{2}(n-1)\beta(1-s)\{n - (n+2)s\}] \\
 &\times \frac{1 + \alpha\{1 - S(\tau)\}\{n - (n+1)s\} + \frac{1}{2}(n-1)\beta(1-s)\{n - (n+2)s\}}{1 + \alpha(1-s)\{n - (n+1)s\} + \frac{1}{2}(n-1)\beta(1-s)\{n - (n+2)s\}} ds. \quad (13.2)
 \end{aligned}$$

Note that  $S_T(\tau)$  is always the same function of  $S(\tau)$ , whatever the functional form of the latter.

The integral in (13.2) can be evaluated in terms of elementary functions (see Appendix), though the expression is, in general, rather complicated. For the special case  $\alpha = 0$  (so that replacement lifetime is independent of the lifetimes of the initial elements),

$$\begin{aligned}
 S_T(\tau) &= \{S(\tau)\}^n [1 + \frac{1}{2}n(n-1)\beta\{1 - S(\tau)\}^2] \\
 &\quad + nS(\tau) \int_{S(\tau)}^1 s^{n-2} [1 + \frac{1}{2}(n-1)\beta(1-s)\{n - (n+2)s\}] ds \\
 &= \frac{S(\tau)[n - \{S(\tau)\}^{n-1}]}{n-1} \\
 &\quad + \frac{\beta S(\tau)}{n+1} [1 - \frac{1}{2}\{S(\tau)\}^{n-1} [n(n+1) - 2(n^2-1)S(\tau) + n(n-1)\{S(\tau)\}^2]] \\
 &= A_n + B_n \beta. \tag{14}
 \end{aligned}$$

Some numerical values of  $A_n$  and  $B_n$  are shown in Table 1.

TABLE 1  
Values of  $A_n$  and  $B_n$  for  $S_T(\tau) = A_n + B_n \beta$  when  $\alpha = 0$ .

$S(\tau) \backslash n$	2	3		4		5		
	$A_n$	$B_n$	$A_n$	$B_n$	$A_n$	$B_n$	$A_n$	$B_n$
0.2	0.360	0.0341	0.296	0.0410	0.266	0.0377	0.250	0.0328
0.4	0.640	0.0288	0.568	0.0475	0.524	0.0546	0.497	0.0547
0.6	0.840	0.0128	0.792	0.0269	0.757	0.0381	0.731	0.0487
0.8	0.960	0.0021	0.944	0.0054	0.930	0.0094	0.918	0.0132

(NOTE: Possible values of  $\beta$  are limited by  $-2n^{-1}(n-1)^{-1} \leq \beta \leq 2n^{-1}$ .)

If  $\alpha \neq 0$ , the value of  $S_T(\tau)$  exceeds that given in (14) by

$$n\alpha S(\tau) \int_{S(\tau)}^1 \frac{s^{n-2} [1 + \frac{1}{2}(n-1)\beta(1-s)\{n - (n+2)s\}] \{s - S(\tau)\} \{n - (n+1)s\}}{1 + \alpha(1-s)\{n - (n+1)s\} + \frac{1}{2}(n-1)\beta(1-s)\{n - (n+2)s\}} ds.$$

If  $S(\tau) > n/(n+1)$ , this will be of opposite sign to  $\alpha$  since

$$\{s - S(\tau)\} \{n - (n+1)s\} < 0 \text{ for } S(\tau) < s < 1.$$

This gives some qualitative appreciation of the effect of non-zero  $\alpha$ .

To see how these results can be used in dealing with more complicated systems, consider the subsystem set out schematically in Figure 1. The

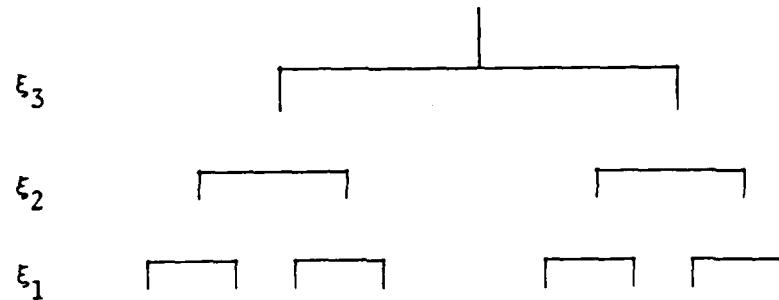


FIGURE 1

component  $\xi_3$  is related to the two  $\xi_2$ 's in the same way as each  $\xi_2$  is related to the two  $\xi_1$ 's belonging to it.  $\xi_3$  comes into operation at the first failure of the two  $\xi_2$ 's. The relevation SDF  $S_T(\tau)$  given by (13.1) now plays the role previously played by the SDF of each  $\xi_1$  ( $S(t)$ ). Of course, if the resulting formula for the SDF of the  $\xi_3$  replacement were to be written out explicitly, it would be very lengthy indeed. However, the computation of numerical values is quite straightforward. It can be carried out in stages: first finding  $S_T(\tau)$  and then combining it with the SDF --  $S_3(t)$ , say -- of the  $\xi_3$  components. (The case of independent components has been treated by Johnson and Kotz (1979).)

### 3. Differential Rates of Aging in Storage and in Service.

It is likely that an element will be under less stress while in storage than in service. This can be allowed for by regarding a period  $t$  in storage as equivalent to a period  $g(t)$  in service (usually with  $g(t) < t$ ). Equation (1)

is then replaced by

$$S_T(\tau) = S_1(\tau) + \int_0^\tau f_1(t) \frac{S_2(g(t) + \tau - t)}{S_2(g(t))} dt, \quad (15)$$

and (2) by

$$S_T(\tau) = S_1(\tau) + \int_0^\tau f_1(t) \frac{S_{2|1}(g(t) + \tau - t | t)}{S_{2|1}(g(t) | t)} dt. \quad (16)$$

One might reasonably take  $g(t) = \gamma t$  -- so that 1 year in storage is equivalent to  $\gamma$  years in service. Usually one would take  $\gamma < 1$ , but this is not essential. Even if it is assumed that  $S_1(t) = S_2(t) = S(t)$ , it is no longer true that  $S_T(\tau)$  depends only on  $S(\tau)$ . As an example, we take  $\gamma < 1$  and

$$S(t) = 1 - \kappa t \quad (0 \leq t \leq \tau \leq \kappa^{-1}).$$

Then from (15),

$$\begin{aligned} S_T(\tau) &= 1 - \kappa\tau + \int_0^\tau \kappa \frac{1 - \kappa\{\tau - (1-\gamma)t\}}{1 - \kappa\gamma t} dt \\ &= 1 - \kappa\tau\gamma^{-1} - (1 - \kappa\tau\gamma)\gamma^{-2} \log(1 - \kappa\tau\gamma). \end{aligned} \quad (17)$$

Some numerical values are given in Table 2.

TABLE 2

Values of  $S_T(\tau)$  when  $S(\tau) = 1 - \kappa\tau$  has specified values  
( $\gamma$  = service years equivalent to one year in storage).

$S(\tau) \backslash \gamma$	0.2	0.4	0.6	0.8	1.0
0.2	0.661	0.639	0.611	0.562(5)	0.522
0.4	0.812	0.804	0.794	0.781	0.767
0.6	0.918	0.915	0.913	0.910	0.906
0.8	0.980	0.979	0.979	0.979	0.979

(The last column corresponds to no difference between aging in storage and in service.) The effect of differential aging is not very great, especially at the longer durations.

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Appendix

The integrand in (13.2) can be expressed in the form

$$[a_n \{(s - b_n)^2 + c_n\}]^{-1} \sum_{j=0}^4 d_{n,j} s^{n+j-2}.$$

Using the change of variable  $y = s - b_n$ , the integral becomes

$$\begin{aligned} & a_n^{-1} \int_{S(\tau) - b_n}^{1-b_n} \left\{ \sum_{j=0}^4 d_{n,j} (y + b_n)^{n-2+j} \right\} (y^2 + c_n)^{-1} dy \\ &= a_n^{-1} \sum_{j=0}^4 d_{n,j} \sum_{h=0}^{n-2+j} \binom{n-2+j}{h} b_n^{n-2+j-h} \int_{S(\tau) - b_n}^{1-b_n} y^h (y^2 + c_n)^{-1} dy \\ &= a_n^{-1} b_n^{n-2} \sum_{j=0}^4 d_{n,j} b_n^j \sum_{h=0}^{n-2+j} \binom{n-2+j}{h} b_n^{-h} [I_h(1 - b_n) - I_h(S(\tau) - b_n)], \end{aligned}$$

where

$$I_h(y) = \int_0^y y^h (y^2 + c_n)^{-1} dy$$

$$= \sum_{i=0}^{[(h-3)/2]} \frac{(-c_n)^i}{h-1-2i} + \begin{cases} \frac{1}{2} (-c_n)^{\frac{1}{2}(h-1)} \log(y^2 + c_n) & (h \text{ odd}) \\ (-1)^{\frac{1}{2}h} c_n^{\frac{1}{2}(h-1)} \tan^{-1}(y/\sqrt{c_n}) & (h \text{ even, } c_n > 0) \\ \frac{1}{2} (-c_n)^{\frac{1}{2}(h-1)} \log \frac{y - \sqrt{-c_n}}{y + \sqrt{-c_n}} & (h \text{ even, } c_n < 0) \end{cases}$$

[For example,  $I_0(y) = 1/\sqrt{c_n} \tan^{-1} y/\sqrt{c_n}$  if  $c_n > 0$ ;  
 $[1/(2\sqrt{-c_n})] \log((y - \sqrt{-c_n})/(y + \sqrt{-c_n}))$  if  $c_n < 0$ ;  
 $I_1(y) = \frac{1}{2} \log(y^2 + c_n)$ ;  
 $I_2(y) = y - c_n I_0(y)$ ;  
 $I_3(y) = \frac{1}{2} y^2 - c_n I_1(y)$ ;  
 $I_4(y) = (1/3)y^3 - c_n I_2(y)$ ; and so on.]

The constants  $a_n$ ,  $b_n$ , and  $c_n$  are:

$$a_n = (n+1)\alpha + \frac{1}{2}(n-1)(n+2)\beta ;$$

$$b_n = \frac{1}{2}\{(2n+1)\alpha + (n^2-1)\beta\}a_n^{-1} ;$$

$$c_n = [a_n - 1/4\{\alpha + (n-1)\beta\}^2]a_n^{-2} .$$

(Note that if  $\alpha = 0$ ,  $a_n = \frac{1}{2}(n-1)(n+2)\beta$ ;  $b_n = (n+1)/(n+2)$ ;  
 $c_n = 2\{(n-1)(n+2)\beta\}^{-1} - (n+2)^{-2}.$ )

The expressions for the coefficients  $d_{n,j}$  are:

$$d_{n,0} = \{1 + \frac{1}{2}n(n-1)\beta\}[1 + n\alpha\{1 - S(\tau)\} + \frac{1}{2}n(n-1)\beta] ;$$

$$d_{n,1} = -2(n^2-1)\{1 + \frac{1}{2}n(n-1)\beta\}\beta - (n+1)\alpha\{1 + (3/2)n(n-1)\beta\}\{1 - S(\tau)\} ;$$

$$d_{n,2} = (n-1)(n+2)\beta + \frac{1}{2}(n-1)(3n^2+6n+2)[\alpha\{1 - S(\tau)\} + (n-1)\beta]\beta ;$$

$$d_{n,3} = -(n^2-1)(n+2)[\frac{1}{2}\alpha\{1 - S(\tau)\} + (n-1)\beta]\beta ;$$

$$d_{n,4} = (1/4)(n-1)^2 (n+2)^2 \beta^2 .$$

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) "Relevation" is the name given by Krakowski (1973) ( <i>Rev. Française Automat. Inform. Recherche Op.</i> , 7, Ser. V-2, 108-120) to the distribution of failure time of a replacement from an aging stock. The present authors (Johnson and Kotz (1979; 1981) ( <i>IEEE Trans. Reliability</i> , R-28, 292-299; <i>American Journal of Mathematical and Management Sciences</i> , 1(2))) have extended this concept to include (i) hierarchical replacement systems and (ii) dependence between lifetimes of original and replacement items. In this paper, we present some further developments, including first steps towards a synthesis of (i) and		

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